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The symbol D , D^2 , ... are called operators. The index of D indicates the number of times the operation of differentiation must be carried out. For example D^3x^4 shows that we must differentiate x^4 three times.

$$\text{Thus, } D^3x^4 = 24x.$$

Negative index of D : D^{-1} is an equivalent to integration. For example

$$D^{-1}x = \int x dx = \frac{x^2}{2}.$$

But it is important to note that the main object of D^{-1} is to find an integral but not the complete integral.

$$D^{-2}(x) = \int [\int x dx] dx = \int \frac{x^2}{2} dx = \frac{x^3}{3 \times 2} = \frac{x^3}{6}.$$

Now, $\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y$ can be written in symbolic form as

$$(D^2 + a_1 D + a_2)y = f(D)y$$

where $f(D)$ or $(D^2 + a_1 D + a_2)$ is operator.

Consider an linear differential eqn. be

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = R \quad (1)$$

(1) can be written also

$$D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \dots + a_n y = R$$

$$\Rightarrow (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) y = R \quad (2)$$

$$\Rightarrow f(D) y = R \quad (3)$$

where $f(D) = D^n + a_1 D^{n-1} + \dots + a_n$

and $f(D)$ now acts as operator and operates on y to yield R .

Eqs. (2) and (3) are called symbolic form of (1).

Auxiliary Equation :— (a.e.) —

we write the a.e. of (3) as $f(D)=0$ and solve it for D .

- Working Rules for finding C.F. (Complementary function)

We first rewrite the given equation in symbolic form like $f(D)y = R$. Then we write auxiliary equation (a.e.) namely $f(D)=0$. Then we solve for D . On solving the a.e. we shall get n roots of n th order equation.

Three cases arise here.

- Simple real roots :— like 2, 5, -1 and so on.
- Complex roots :— of the type $\alpha \pm i\beta$
- Roots of the type :— $\alpha \pm \sqrt{\beta}$

we discuss one by one.

Case I: — First suppose that the a.e. has n distinct roots say $m_1, m_2, m_3, \dots, m_n$ then the C.F. is given by

$$C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + \dots + C_n e^{m_n x}$$

where $C_1, C_2, C_3, \dots, C_n$ are arbitrary constants.

For example — If the roots be 2 and 3, then

$$\text{C.F.} = C_1 e^{2x} + C_2 e^{3x}$$

Again, if the a.e. has the real roots m occurring k times and further the remaining roots of the a.e. are distinct real nos.

$m_{k+1}, m_{k+2}, \dots, m_n$. Then C.F. is given by

$$(C_1 + C_2 x + C_3 x^2 + \dots + C_k x^{k-1}) e^{mx} + C_{k+1} x e^{m k+1 x} + \dots$$

For example : —

If the roots be 2, 2, 2, -1, 3 then

$$\text{C.F.} = (C_1 + C_2 x + C_3 x^2) e^{3x} + C_4 e^{-x} + C_5 e^{3x}$$

Case II: — Let $\alpha \pm i\beta$ be a pair of complex roots. Then the corresponding part of C.F. may be written in one of the following forms.

$$(i) e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x).$$

$$(ii) C_1 e^{\alpha x} \cos (\beta x + C_2).$$

$$(iii) C_1 e^{\alpha x} \sin (\beta x + C_2).$$

If $\alpha \pm i\beta$ occur twice then C.F. is written as

$$e^{\alpha x} \{ (c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x \}$$

Case III: — If a pair of the roots of the a.e. be like as $\alpha \pm \sqrt{\beta}$, where β is positive, then corresponding part of C.F. is one of the following three forms —

$$(i) e^{\alpha x} [c_1 \cosh(\alpha \sqrt{\beta}) + c_2 \sinh(\alpha \sqrt{\beta})].$$

$$(ii) c_1 e^{\alpha x} \cosh(\alpha \sqrt{\beta} + c_2).$$

$$(iii) c_1 e^{\alpha x} \sinh(\alpha \sqrt{\beta} + c_2).$$

[For P.I. (Particular Integral)]

Case I: — Working rule for finding P.I. when R (i.e. right hand side of the given equation) is of the form e^{ax}

$$(i) P.I. = \frac{1}{f(D)} \cdot e^{ax} = \frac{1}{f(a)} \cdot e^{ax} \quad \text{where } f(a) \neq 0.$$

$$\text{e.g.; P.I.} = \frac{1}{D^2 + D + 5} \cdot e^{-2x} = \frac{1}{(-2)^2 - 2 + 5} \cdot e^{-2x} = \frac{e^{-2x}}{7}$$

(ii) If $f(a) = 0$ then $f(D)$ must possess a factor of the type $(D-a)^r$

$$\frac{1}{(D-a)^r} \cdot e^{ax} = \frac{x^r}{r!} \cdot e^{ax}$$

(iii) If R be a constant say 5, then

$$\text{P.I.} = \frac{1}{D^2 + D + 1} \cdot 5 = \frac{1}{D^2 + D + 1} \cdot 5 \cdot e^{0 \cdot x}$$

$$= 5 \cdot \frac{1}{D^2 + D + 1} \cdot e^{0 \cdot x} = 5 \cdot \frac{1}{0+0+1} \cdot e^{0 \cdot x} = 5$$

Case-2: — (a) Working rule for P.I. when R is $\cos(ax)$ or $\sin(ax)$

$$(i) \text{ P.I.} = \frac{1}{f(D)} \cdot \cos(ax) = \frac{1}{\phi(D^2)} \cdot \cos(ax)$$

$$= \frac{1}{\phi(-a^2)} \cdot \cos(ax).$$

$$\text{e.g., } \text{P.I.} = \frac{1}{D^4 + D^2 + 1} \cdot \cos 2x = \frac{1}{(D^2)^2 + D^2 + 1} \cdot \cos 2x$$

$$= \frac{1}{(-2^2)^2 + (-2^2) + 1} \cdot \cos 2x = \frac{1}{13} \cdot \cos 2x$$

(ii) Sometime, we can not form $\phi(D^2)$. In such case we proceed as follows: —

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1} \cdot \cos 3x = \frac{1}{-3^2 - 2D + 1} \cdot \cos 3x = \frac{1}{-9 - 2D + 1} \cdot \cos 3x$$

$$= -\frac{1}{2} \cdot \frac{1}{D+4} \cdot \cos 3x = -\frac{1}{2} \cdot \frac{D-4}{D^2 - 4^2} \cdot \cos 3x$$

$$= -\frac{1}{2} (D-4) \cdot \frac{1}{-3^2 - 4^2} \cdot \cos 3x$$

$$= \frac{1}{50} (D-4) \cos 3x$$

$$= \frac{1}{50} [D(\cos 3x) - 4 \cos 3x]$$

$$= \frac{1}{50} [-(\sin 3x) \cdot 3 - 4 \cos 3x]$$

(b) If $\phi(-a^2) = 0$ then we shall use the following formula:

$$(i) \frac{1}{D^2+a^2} \sin(ax) = -\frac{x}{2a} \cdot \cos(ax)$$

$$(ii) \frac{1}{D^2+a^2} \cdot \cos(ax) = \frac{x}{2a} \sin(ax)$$

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